



# Conductors in $p$ -adic families

Jyoti Prakash Saha<sup>1</sup> 

Received: 28 November 2015 / Accepted: 10 August 2016

© The Author(s) 2016. This article is published with open access at Springerlink.com

**Abstract** Given a Weil-Deligne representation of the Weil group of an  $\ell$ -adic number field with coefficients in a domain  $\mathcal{O}$ , we show that its pure specializations have the same conductor. More generally, we prove that the conductors of a collection of pure representations are equal if they lift to Weil-Deligne representations over domains containing  $\mathcal{O}$  and the traces of these lifts are parametrized by a pseudorepresentation over  $\mathcal{O}$ .

**Keywords**  $p$ -adic families of automorphic forms · Pure representations · Conductors

**Mathematics Subject Classification** 11F55 · 11F80

## 1 Introduction

The aim of this article is to study the variation of automorphic and Galois conductors in  $p$ -adic families of automorphic Galois representations, for instance, in Hida families and eigenvarieties. We relate the variation of Galois conductors in families to the purity of  $p$ -adic automorphic Galois representations at the finite places not dividing  $p$  and for the variation of automorphic conductors, we use local-global compatibility. We establish the constancy of tame conductors at arithmetic points lying along irreducible components of  $p$ -adic families.

---

✉ Jyoti Prakash Saha  
saha@mpim-bonn.mpg.de

<sup>1</sup> Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

## 1.1 Motivation

In [9, 10], Hida showed that the  $p$ -ordinary eigen cusp forms, i.e., the normalized eigen cusp forms whose  $p$ -th Fourier coefficients are  $p$ -adic units (with respect to fixed embeddings of  $\overline{\mathbb{Q}}$  in  $\mathbb{C}$  and  $\overline{\mathbb{Q}}_p$ ), can be put in  $p$ -adic families. More precisely, he showed that for each positive integer  $N$  and an odd prime  $p$  with  $p \nmid N$  and  $Np \geq 4$ , there is a subset of the set of  $\overline{\mathbb{Q}}_p$ -specializations of the universal  $p$ -ordinary Hecke algebra  $h^{\text{ord}}(N; \mathbb{Z}_p)$ , called the set of arithmetic specializations, such that there is a one-to-one correspondence between the arithmetic specializations of  $h^{\text{ord}}(N; \mathbb{Z}_p)$  and the  $p$ -ordinary  $p$ -stabilized normalized eigen cusp forms of tame level a divisor of  $N$ . It turns out that the tame conductors of the Galois representations attached to the arithmetic specializations remain constant along irreducible components of  $h^{\text{ord}}(N; \mathbb{Z}_p)$  (see [11, Theorem 4.1]). Following Hida's construction of families of ordinary cusp forms, further examples of families of automorphic Galois representations are constructed, for instance, Hida families of ordinary automorphic representations of definite unitary groups, families of overconvergent forms (see the works of Hida [12], Coleman and Mazur [6], Chenevier [4], Bellaïche and Chenevier [1] et al.). The aim of this article is to understand the variation of automorphic and Galois conductors in these families of automorphic Galois representations. In many cases, the restrictions of  $p$ -adic automorphic Galois representations to decomposition groups at places outside  $p$  are known to be pure (i.e. the monodromy filtration and the weight filtration associated to such a local Galois representation are equal up to some shift by an integer, see [18, p. 471], [15, p. 1014] for details). So we focus on the variation of conductors in families of pure representations of local Galois groups.

## 1.2 Results obtained

Let  $p, \ell$  be two distinct primes and  $K$  be a finite extension of  $\mathbb{Q}_\ell$ . Let  $\mathcal{O}$  be an integral domain containing  $\mathbb{Q}$ . In Theorem 3.1, we show that given any Weil-Deligne representation of the Weil group  $W_K$  of  $K$  with coefficients in  $\mathcal{O}$ , its conductor coincides with the conductors of its pure specializations over  $\overline{\mathbb{Q}}_p$ . Next, in Theorem 3.2, we show that a collection of pure representations of  $W_K$  over  $\overline{\mathbb{Q}}_p$  have the same conductor if they lift to Weil-Deligne representations of  $W_K$  over domains containing  $\mathcal{O}$  and the traces of these lifts are parametrized by a pseudorepresentation  $T : W_K \rightarrow \mathcal{O}$ .

An eigenvariety is an example of a family of Galois representations. The traces of the Galois representations associated to the arithmetic points of an eigenvariety are interpolated by a pseudorepresentation defined over its global sections. By [2, Lemma 7.8.11], this pseudorepresentation lifts to a Galois representation on a finite type module over some integral extension of the normalization of  $\mathcal{O}(U)$  for any non-empty admissible open affinoid subset  $U$ . But this module is not known to be free. So Theorem 3.2 cannot be used to study the tame conductors of all arithmetic points. To circumvent this problem, we establish Theorem 3.3 which can be used to study the tame conductors of a large class of arithmetic points, for example, if the associated Galois representations are absolutely irreducible and yield pure representations when restricted to decomposition groups at the places outside  $p$ . Theorems 3.1, 3.2, and 3.3

apply to  $p$ -adic families of automorphic representations, for instance, to ordinary families, overconvergent families, and explain the variation of the tame conductors. We illustrate it using the example of Hida family of ordinary automorphic representations for definite unitary groups (see Theorem 4.1).

### 1.3 Notations

For each field  $F$ , fix an algebraic closure  $\overline{F}$  of it and denote the Galois group  $\text{Gal}(\overline{F}/F)$  by  $G_F$ . The decomposition group  $\text{Gal}(\overline{E}_v/E_v)$  of a number field  $E$  at a finite place  $v$  is denoted by  $G_v$ . Let  $W_v \subset G_v$  (resp.  $I_v \subset G_v$ ) denote the Weil group (resp. inertia group) and  $\text{Fr}_v \in G_v/I_v$  denote the geometric Frobenius element. The fraction field of an integral domain  $R$  is denoted by  $Q(R)$ , the field  $\overline{Q(R)}$  is denoted by  $\overline{Q(R)}$  and the integral closure of  $R$  in  $\overline{Q(R)}$  is denoted by  $R^{\text{intal}}$ . For each map  $f : R \rightarrow S$  between domains, we fix an extension  $f^{\text{intal}} : R^{\text{intal}} \rightarrow S^{\text{intal}}$  of  $f$ .

Denote by  $q$  the cardinality of the residue field  $k$  of the ring of integers  $\mathcal{O}_K$  of  $K$ . Let  $I_K$  denote the inertia subgroup of  $G_K$ . Let  $\{G_K^s\}_{s \geq -1}$  denote the upper numbering filtration on  $G_K$  by ramification subgroups. The Weil group  $W_K$  is defined as the subgroup of  $G_K$  consisting of elements which map to an integral power of the geometric Frobenius element  $\text{Fr}_k$  in  $G_k$ . Its topology is determined by decreeing that  $I_K$  with its usual topology is an open subgroup. Define  $v_K : W_K \rightarrow \mathbb{Z}$  by  $\sigma|_{K^{\text{ur}}} = \text{Fr}_k^{v_K(\sigma)}$  for all  $\sigma \in W_K$ . Let  $\phi$  be an element of  $G_K$  which maps to  $\text{Fr}_k \in G_k$ .

## 2 Preliminaries

In this section, we provide the definition of Weil-Deligne representations, conductors, etc.

**Definition 2.1** ([7, 8.4.1]) Let  $M$  be a free module of finite rank over a commutative domain  $A$  of characteristic zero. A *Weil-Deligne representation* of  $W_K$  on  $M$  is a pair  $(r, N)$  where  $r : W_K \rightarrow \text{Aut}_A(M)$  is a representation with open kernel and  $N$  is a nilpotent element of  $\text{End}_A(M)$  such that

$$r(\sigma)Nr(\sigma)^{-1} = q^{-v_K(\sigma)}N$$

holds for any  $\sigma \in W_K$ .

Let  $(r, N)$  be a Weil-Deligne representation on a vector space  $V$  with coefficients in a field containing the characteristic roots of all elements of  $r(W_K)$ . Let  $r(\phi) = r(\phi)^{\text{ss}}u = ur(\phi)^{\text{ss}}$  be the Jordan decomposition of  $r(\phi)$  as the product of a diagonalizable operator  $r(\phi)^{\text{ss}}$  and a unipotent operator  $u$  on  $V$ . Following [7, 8.5], define  $\tilde{r}(\sigma) = r(\sigma)u^{-v_K(\sigma)}$  for all  $\sigma \in W_K$ . Then the pair  $(\tilde{r}, N)$  is also a Weil-Deligne representation on  $V$  (by [7, 8.5]). It is called the *Frobenius-semisimplification* of  $(r, N)$  (cf. [7, 8.6]) and it is denoted by  $(r, N)^{\text{Fr-ss}}$ .

**Definition 2.2** If  $(r, N)$  is Weil-Deligne representation of  $W_K$  on a vector space  $V$  over an algebraically closed field of characteristic zero, then its *conductor* is defined as

$$\text{cond}(r, N) = \text{codim } V^{I_K, N=0} + \int_0^\infty \text{codim } V^{G_K^u} du$$

where  $V^{I_K, N=0}$  denotes the subspace of  $V$  on which  $I_K$  acts trivially and  $N$  is zero.

**Lemma 2.3** Let  $f : A \rightarrow B$  be a map between domains of characteristic zero. Let  $E$  (resp.  $F$ ) be a field containing  $A$  (resp.  $B$ ). Let  $\rho : G \rightarrow \text{GL}_n(A)$  be a representation of a group  $G$ . Then the  $E$ -dimension of the space of  $G$ -invariants of the representation  $G \xrightarrow{\rho} \text{GL}_n(A) \hookrightarrow \text{GL}_n(E)$  and the  $F$ -dimension of the space of  $G$ -invariants of the representation  $G \xrightarrow{\rho} \text{GL}_n(A) \xrightarrow{f} \text{GL}_n(B) \hookrightarrow \text{GL}_n(F)$  are equal if  $G$  is finite.

*Proof* If  $r : G \rightarrow \text{GL}_n(\mathcal{F})$  is a representation where  $\mathcal{F}$  is a field of characteristic zero, then the dimension of the space of  $G$ -invariants of  $r$  is equal to the trace of the idempotent operator  $\frac{1}{|G|} \sum_{g \in G} r(g)$ . Since  $f(m)$  is equal to  $m$  for any integer  $m$ , the lemma follows.  $\square$

### 3 Main results

Denote the fraction field of  $\mathcal{O}$  by  $\mathcal{L}$  and the algebraic closure of  $\mathbb{Q}$  in  $\mathcal{O}$  by  $\mathbb{Q}^{\text{cl}}$ .

**Theorem 3.1** Let  $(r, N) : W_K \rightarrow \text{GL}_n(\mathcal{O})$  be a Weil-Deligne representation of  $W_K$  with coefficients in  $\mathcal{O}$ . If  $f : \mathcal{O} \rightarrow \mathbb{Q}_p$  is a map such that  $f \circ (r, N)$  is pure, then the conductor of  $f \circ (r, N)$  is equal to the conductor of  $(r, N) \otimes_{\mathcal{O}} \mathcal{L}$ .

*Proof* Since  $f \circ (r, N)$  is pure, by [14, Theorem 4.1], there exist positive integers  $m, t_1, \dots, t_m$ , unramified characters  $\chi_1, \dots, \chi_m : W_K \rightarrow (\mathcal{O}^{\text{intal}})^{\times}$ , representations  $\rho_1, \dots, \rho_m$  of  $W_K$  over  $\mathbb{Q}^{\text{cl}}$  with finite image such that

$$\begin{aligned} \left( (r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}} \right)^{\text{Fr-ss}} &\simeq \bigoplus_{i=1}^n \text{Sp}_{t_i}(\chi_i \otimes \rho_i)_{/\overline{\mathcal{L}}}, \\ (f \circ (r, N))^{\text{Fr-ss}} &\simeq \bigoplus_{i=1}^n \text{Sp}_{t_i} \left( f^{\text{intal}} \circ (\chi_i \otimes \rho_i) \right)_{/\overline{\mathbb{Q}}_p}. \end{aligned}$$

So the dimension of  $((r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}})^{I_K, N=0}$  (resp.  $(f \circ (r, N))^{I_K, N=0}$ ) over  $\overline{\mathcal{L}}$  (resp.  $\overline{\mathbb{Q}}_p$ ) is equal to  $\sum_{i=1}^n \rho_i^{I_K}$  (resp.  $\sum_{i=1}^n (f^{\text{intal}} \circ \rho_i)^{I_K}$ ). By Lemma 2.3, the  $\mathcal{L}$ -dimension of  $((r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}})^{I_K, N=0}$  is equal to the  $\overline{\mathbb{Q}}_p$ -dimension of  $(f \circ (r, N))^{I_K, N=0}$ . Note that  $G_K^u$  is contained in  $I_K$  for any  $u \geq 0$  and  $r(H)$  is finite for any subgroup  $H$  of  $I_K$ . Hence  $\dim_{\overline{\mathcal{L}}}((r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}})^H$  is equal to  $\dim_{\overline{\mathbb{Q}}_p}(f \circ (r, N))^H$  by Lemma 2.3. This shows that the conductor of  $f \circ (r, N)$  is equal to the conductor of  $(r, N) \otimes_{\mathcal{O}} \overline{\mathcal{L}}$ .  $\square$

Now we establish an analogue of the above result for pseudorepresentations of Weil groups. We refer to [17, Section 1] for the definition of pseudorepresentation. They are defined by abstracting the crucial properties of the trace of a group representation.

**Theorem 3.2** *Let  $\mathcal{O}$  be an integral domain and  $\text{res} : \mathcal{O} \hookrightarrow \mathcal{O}$  be an injective map. Let  $T : W_K \rightarrow \mathcal{O}$  be a pseudorepresentation and let  $(r, N) : W_K \rightarrow \text{GL}_n(\mathcal{O})$  be a Weil-Deligne representation such that  $\text{res} \circ T = \text{trr}$  and  $f \circ (r, N)$  is pure for some map  $f : \mathcal{O} \rightarrow \overline{\mathbb{Q}}_p$ . Then there exists a non-negative integer  $C$  such that the following statement holds. If  $(r', N') : W_K \rightarrow \text{GL}_n(\mathcal{O}')$  is a Weil-Deligne representation over a domain  $\mathcal{O}'$  such that  $\text{res}' \circ T = \text{trr}'$  for some injective map  $\text{res}' : \mathcal{O} \hookrightarrow \mathcal{O}'$  and  $f' \circ (r', N')$  is pure for some map  $f' : \mathcal{O}' \rightarrow \overline{\mathbb{Q}}_p$ , then*

$$\text{cond}((r', N') \otimes_{\mathcal{O}'} \overline{Q}(\mathcal{O}')) = \text{cond}(f' \circ (r', N')) = C.$$

*Proof* By [14, Theorem 5.4, Proposition 2.8], there exist positive integers  $m, t_1, \dots, t_m$ , unramified characters  $\chi_1, \dots, \chi_m : W_K \rightarrow (\mathcal{O}^{\text{intal}})^{\times}$ , representations  $\rho_1, \dots, \rho_m$  of  $W_K$  over  $\mathbb{Q}^{\text{cl}}$  with finite image such that  $((r', N') \otimes_{\mathcal{O}'} \overline{Q}(\mathcal{O}'))^{\text{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\text{res}'^{\text{intal}} \circ (\chi_i \otimes \rho_i))$  for any Weil-Deligne representation  $(r', N') : W_K \rightarrow \text{GL}_n(\mathcal{O}')$  as above. Then Lemma 2.3 shows that the conductors of  $(r', N') \otimes_{\mathcal{O}'} \overline{Q}(\mathcal{O}')$  and  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\chi_i \otimes \rho_i)$  are equal. So by Theorem 3.1, the result follows by taking  $C$  to be the conductor of  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\chi_i \otimes \rho_i)$ .  $\square$

Let  $w$  be a finite place of a number field  $F$  not dividing  $p$  and assume that  $\mathcal{O}$  is a  $\mathbb{Z}_p$ -algebra. Suppose  $T, T_1, \dots, T_n$  are pseudorepresentations of  $G_F$  with values in  $\mathcal{O}$  such that  $T$  is equal to  $T_1 + \dots + T_n$ . By [17, Theorem 1], there exist semisimple representations  $\sigma_1, \dots, \sigma_n$  of  $G_F$  over  $\overline{\mathcal{L}}$  such that  $\text{tr}\sigma_i = T_i$  for all  $1 \leq i \leq n$ .

**Theorem 3.3** *Suppose the actions of the inertia subgroup  $I_w$  on the representations  $\sigma_1, \dots, \sigma_n$  are potentially unipotent. Let  $(\mathcal{O}, \mathfrak{m}, \kappa, \text{loc}, \rho_1, \dots, \rho_n)$  be a tuple where  $\mathcal{O}$  is a  $\mathbb{Z}_p$ -algebra, it is a Henselian Hausdorff domain with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa$  which is an algebraic extension of  $\mathbb{Q}_p$ ,  $\text{loc} : \mathcal{O} \hookrightarrow \mathcal{O}$  is an injective map of  $\mathbb{Z}_p$ -algebras and for any  $1 \leq i \leq n$ ,  $\rho_i$  is an irreducible  $G_F$ -representation over  $\overline{\kappa}$  such that  $\rho_i|_{G_w}$  is pure and  $\text{tr}\rho_i$  is equal to  $\text{loc} \circ T_i \bmod \mathfrak{m}$ . Then we have*

$$\text{cond}\left(\text{WD}\left(\bigoplus_{i=1}^n \rho_i|_{W_w}\right)\right) = \text{cond}\left(\text{WD}\left(\bigoplus_{i=1}^n \sigma_i|_{W_w}\right)\right).$$

*Proof* By [14, Theorem 5.6, Proposition 2.8], there exist positive integers  $m, t_1, \dots, t_m$ , unramified characters  $\chi_1, \dots, \chi_m : W_K \rightarrow (\mathcal{O}^{\text{intal}})^{\times}$ , representations  $\theta_1, \dots, \theta_m$  of  $W_K$  over  $\mathbb{Q}^{\text{cl}}$  with finite image such that  $\text{WD}(\bigoplus_{i=1}^n \rho_i|_{W_w})^{\text{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\pi_{\mathfrak{m}}^{\text{intal}} \circ \text{loc}^{\text{intal}} \circ (\chi_i \otimes \theta_i))$  and  $\text{WD}(\bigoplus_{i=1}^n \sigma_i|_{W_w})^{\text{Fr-ss}}$  is isomorphic to  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\chi_i \otimes \theta_i)$  where  $\pi_{\mathfrak{m}}$  denotes the mod  $\mathfrak{m}$  reduction map  $\mathcal{O} \rightarrow \mathcal{O}/\mathfrak{m}$ . Then Lemma 2.3 shows that the conductors of  $\text{WD}(\bigoplus_{i=1}^n \rho_i|_{W_w})$  and  $\text{WD}(\bigoplus_{i=1}^n \sigma_i|_{W_w})$  are equal to the conductor of  $\bigoplus_{i=1}^m \text{Sp}_{t_i}(\chi_i \otimes \theta_i)$ . The result follows.  $\square$

## 4 Conductors in families

In this section, using the example of a Hida family of ordinary automorphic representations for definite unitary groups, we show how results of Sect. 3 describes the variation of tame conductors in  $p$ -adic families.

Let  $F^+$  denote the maximal totally real subfield of a CM field  $F$ . Let  $n \geq 2$  be an integer such that  $n[F^+ : \mathbb{Q}]$  is divisible by 4 if  $n$  is even. Let  $\ell > n$  be a rational prime such that every prime of  $F^+$  dividing  $\ell$  splits in  $F$ . Let  $K$  be a finite extension of  $\mathbb{Q}_\ell$  in  $\overline{\mathbb{Q}_\ell}$  containing the image of every embedding of  $F$  in  $\overline{\mathbb{Q}_\ell}$ . Denote by  $S_\ell$  the set of places of  $F^+$  dividing  $\ell$ . In the following,  $R$  denotes a finite set of finite places of  $F^+$  which split in  $F$ . Assume that  $R$  is disjoint from  $S_\ell$ . For each  $v \in S_\ell \cup R$ , fix a place  $\tilde{v}$  of  $F$  above  $v$ . For each place  $v \in R$ , let  $\text{Iw}(\tilde{v})$  denote the compact open subgroup of  $\text{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$  as in [8, Section 2.1] and  $\chi_v$  denote the character as in section 2.2 of *loc. cit.*

Let  $G$  be the reductive algebraic group over  $F^+$  as in [8, Section 2.1]. For any dominant weight  $\lambda$  for  $G$  (see [8, Definition 2.2.3]), each of the spaces  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}_\ell})$ ,  $S_{\lambda, \{\chi_v\}}^{\text{ord}}(\mathcal{O}_K)$  has an action of the group  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\tilde{v})$  (see [8, Definitions 2.2.4, 2.4.2]). The weak base change of an irreducible constituent  $\pi$  of the  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\tilde{v})$ -representation  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}_\ell})$  to  $\text{GL}_n(\mathbb{A}_F)$  is denoted by  $\text{WBC}(\pi)$ , whose existence follows from [13, Corollaire 5.3]. By [5, Theorem 3.2.3], there exists a unique (up to equivalence) continuous semisimple representation  $r_\pi : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}_\ell})$  (as in [8, Proposition 2.7.2]) associated to  $\text{WBC}(\pi)$ .

An *ordinary automorphic representation* for  $G$  is an irreducible constituent  $\pi$  of the  $G(\mathbb{A}_{F^+}^{\infty, R}) \times \prod_{v \in R} \text{Iw}(\tilde{v})$ -representation  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}_\ell})$  such that the space  $\pi^{U(b,c)} \cap S_{\lambda, \{\chi_v\}}^{\text{ord}}(U(b,c), \mathcal{O}_K)$  is nonzero for some integers  $0 \leq b \leq c$  (see [8, Definition 2.2.4, Section 2.3]). Let  $T$  be a finite set of finite places of  $F^+$  such that every place in  $T$  splits in  $F$  and  $R \cup S_\ell \subseteq T$  (see [8, §2.3]). Denote the universal ordinary Hecke algebra  $\mathbb{T}_{\{\chi_v\}}^{T, \text{ord}}(U(\infty), \mathcal{O}_K)$  (as in [8, Definition 2.6.2]) by  $\mathbb{T}^{\text{ord}}$  where  $U$  is a compact open subgroup of  $G(\mathbb{A}_{F^+}^{\infty})$ . The ring  $\mathbb{T}^{\text{ord}}$  has a  $\Lambda$ -algebra structure and is finite over  $\Lambda$  where  $\Lambda$  denotes the completed group algebra as in [8, Definition 2.5.1]. If  $A$  is a finite  $\Lambda$ -algebra, then a map  $f : A \rightarrow \overline{\mathbb{Q}_\ell}$  of  $\mathcal{O}_K$ -algebras is said to be an *arithmetic specialization* of  $A$  if  $\ker(f|_\Lambda)$  is equal to the prime ideal  $\wp_{\lambda, \alpha}$  (see [8, Definition 2.6.3]) of  $\Lambda$  for some dominant weight  $\lambda$  for  $G$  and a finite order character  $\alpha : T_n(\mathfrak{l}) \rightarrow \mathcal{O}_K^\times$ . Each arithmetic specialization  $\eta$  of  $\mathbb{T}^{\text{ord}}$  determines an ordinary automorphic representation  $\pi_\eta$  for  $G$  by [8, Lemma 2.6.4].

For a non-Eisenstein maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}^{\text{ord}}$  (in the sense of [8, Section 2.7]), let  $r_\mathfrak{m}$  denote the representation of  $G_{F^+}$  as in [8, Proposition 2.7.4]. Composing the restriction of  $r_\mathfrak{m}$  to  $G_F$  with the projection map  $\text{GL}_n(\mathbb{T}_\mathfrak{m}^{\text{ord}}) \times \text{GL}_1(\mathbb{T}_\mathfrak{m}^{\text{ord}}) \rightarrow \text{GL}_n(\mathbb{T}_\mathfrak{m}^{\text{ord}})$ , we obtain a continuous representation  $G_F \rightarrow \text{GL}_n(\mathbb{T}_\mathfrak{m}^{\text{ord}})$  which we denote by  $r_\mathfrak{m}$  by abuse of notation. Since  $\mathfrak{m}$  is non-Eisenstein, by [8, Propositions 2.7.2, 2.7.4], the  $G_F$ -representations  $\eta \circ r_\mathfrak{m}, r_{\pi_\eta}$  are isomorphic for any arithmetic specialization  $\eta$  of  $\mathbb{T}_\mathfrak{m}^{\text{ord}}$ .

**Theorem 4.1** *Let  $\mathfrak{a}$  be a minimal prime of  $\mathbb{T}^{\text{ord}}$  contained in a non-Eisenstein maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}^{\text{ord}}$ . Let  $w$  be a finite place of  $F$  not dividing  $\ell$ . Then there exists a non-*

negative integer  $C$  such that the conductor of  $\mathrm{WD}(r_{\pi_{\eta}}|_{w_w})$  is equal to  $C$  for any arithmetic specialization  $\eta$  of  $\mathbb{T}_{\mathfrak{m}}^{\mathrm{ord}}/\mathfrak{a}$  for which the weak base change  $\mathrm{WBC}(\pi_{\eta})$  of  $\pi_{\eta}$  is cuspidal. Moreover, the conductor of  $\mathrm{WBC}(\pi_{\eta})_w$  is also equal to  $C$  for any such specialization  $\eta$ .

**Proof** If the weak base change of an irreducible constituent  $\pi$  of the  $G(\mathbb{A}_{F+}^{\infty,R}) \times \prod_{v \in R} \mathrm{Iw}(\tilde{v})$ -representation  $S_{\lambda, \{\chi_v\}}(\overline{\mathbb{Q}}_{\ell})$  is cuspidal, then from [3, Theorems 1.1, 1.2], the proofs of theorem 5.8 and corollary 5.9 of *loc. cit.*, it follows that the representation  $r_{\pi}|_{G_w}$  is pure for any finite place  $w$  of  $F$  not dividing  $\ell$ . Note that the action of the inertia subgroup  $I_w$  on the representation  $r_{\mathfrak{m}} \bmod \mathfrak{a}$  is potentially unipotent by Grothendieck's monodromy theorem (see [16, pp. 515–516]). So Theorem 3.1 gives the first part. Since local Langlands correspondence preserves conductors, the rest follows from [3, Theorem 1.1] on local-global compatibility of cuspidal automorphic representations for  $\mathrm{GL}_n$ .  $\square$

**Acknowledgments** It is a pleasure to thank Olivier Fouquet for his guidance during the preparation of this article. I also thank the anonymous referee for providing helpful comments. I am grateful to the Max Planck Institute for Mathematics for providing funding for the open access publication. I thank the Mathematisches Forschungsinstitut Oberwolfach and the Max Planck Institute for Mathematics for providing financial support.

**Open Access** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

## References

1. Bellaïche, J., Chenevier, G.: Formes non tempérées pour  $\mathrm{U}(3)$  et conjectures de Bloch-Kato. *Ann. Sci. École Norm. Sup.* **37**(4), 611–662 (2004)
2. Bellaïche, J., Chenevier, G.: Families of Galois representations and Selmer groups. *Astérisque* **324**, 1–314 (2009)
3. Caraiani, A.: Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.* **161**(12), 2311–2413 (2012)
4. Chenevier, G.: Familles  $p$ -adiques de formes automorphes pour  $\mathrm{GL}_n$ . *J. Reine Angew. Math.* **570**, 143–217 (2004)
5. Chenevier, G., Harris, M.: Construction of automorphic Galois representations. II. *Camb. J. Math.* **1**(1), 53–73 (2013)
6. Coleman, R., Mazur, B.: The eigencurve. In: *Galois Representations in Arithmetic Algebraic geometry* (Durham, 1996). London Mathematical Society Lecture Note Series, vol. 254, pp. 1–113. Cambridge University Press, Cambridge (1998)
7. Deligne, P.: Les constantes des équations fonctionnelles des fonctions  $L$ . In: *Modular Functions of One Variable, II* (Proceedings International Summer School, University of Antwerp, Antwerp, 1972. Lecture Notes in Mathematics), vol. 349, pp. 501–597. Springer, Berlin (1973)
8. Geraghty, D.J.: Modularity lifting theorems for ordinary Galois representations. Ph.D. thesis, Harvard University, ProQuest LLC, Ann Arbor (2010)
9. Hida, H.: Galois representations into  $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$  attached to ordinary cusp forms. *Invent. Math.* **85**(3), 545–613 (1986)
10. Hida, H.: Iwasawa modules attached to congruences of cusp forms. *Ann. Sci. École Norm. Sup.* **19**(2), 231–273 (1986)
11. Hida, H.: A  $p$ -adic measure attached to the zeta functions associated with two elliptic modular forms. II. *Ann. Inst. Fourier (Grenoble)* **38**(3), 1–83 (1988)

12. Hida, H.: Control theorems of  $p$ -nearly ordinary cohomology groups for  $SL(n)$ . *Bull. Soc. Math. Fr.* **123**(3), 425–475 (1995)
13. Labesse, J.-P.: Changement de base CM et séries discrètes. In: *On the Stabilization of the Trace Formula (Stabilization of the Trace Formula, Shimura Varieties, and Arithmetic Applications, vol. 1)*, pp. 429–470. International Press, Somerville (2011)
14. Saha, J.P.: Purity for families of Galois representations. <http://arxiv.org/abs/1410.3843v2.pdf> (2014)
15. Scholl, A.J.: Hypersurfaces and the Weil conjectures. *Int. Math. Res. Not.* **5**, 1010–1022 (2011)
16. Serre, J.P., Tate, J.: Good reduction of abelian varieties. *Ann. Math.* **2**(88), 492–517 (1968)
17. Taylor, R.: Galois representations associated to Siegel modular forms of low weight. *Duke Math. J.* **63**(2), 281–332 (1991)
18. Taylor, R., Yoshida, T.: Compatibility of local and global Langlands correspondences. *J. Am. Math. Soc.* **20**(2), 467–493 (2007)